Perfect type of n-tensors

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Abstract

In various application fields, tensor type data are used recently and then a typical rank is important. Although there may be more than one typical ranks over the real number field, a generic rank over the complex number field is the minimum number of them. The set of n-tensors of type $p_1 \times p_2 \times \cdots \times p_n$ is called perfect, if it has a typical rank $\max(p_1, \dots, p_n)$. In this paper, we determine perfect types of n-tensor.

1 Introduction

An $p_1 \times p_2 \times \cdots \times p_n$ tensor over a field $\mathbb{F}$ is an element of the tensor product of n vector spaces $\mathbb{F}^{p_1}, \mathbb{F}^{p_2}, \dots, \mathbb{F}^{p_n}$. Thus every tensor can be expressed as a sum of tensors of the form $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_n$ for $\boldsymbol{a}_i \in \mathbb{F}^{p_i}$, $i = 1, 2, \dots, n$. The rank $\operatorname{rank}_{\mathbb{F}} T$ of a tensor T means that the minimum number r of rank one tensors which express T as a sum. The rank depends on the field.

The set $T(p_1, \dots, p_n; \mathbb{F})$ of all $p_1 \times \cdots \times p_n$ tensors is $\mathbb{F}^{p_1} \times \cdots \times \mathbb{F}^{p_n}$ as a set. We consider the Euclidean topology on $\mathbb{F}^{p_1} \times \cdots \times \mathbb{F}^{p_n} = \mathbb{F}^{p_1 \cdots p_n}$ as a topology on the set $T(p_1, \dots, p_n; \mathbb{F})$.

Now let $\mathbb{F}$ be the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. A typical rank, denoted by $\text{typical\_rank}_{\mathbb{F}}(p_1, \dots, p_n)$, of $T(p_1, \dots, p_n; \mathbb{F})$ is defined as the set of integers r such that the set of rank r tensors has a positive Lebesgue measure in $T(p_1, \dots, p_n; \mathbb{F})$. A typical rank of tensors is one of important tools for experimental simulation. We know a typical rank of 3-tensors of special types. ten Berge obtained that the typical rank of $m \times n \times 2$ tensors is $\min(n, 2m)$ if $2 \le m < n$ and $\{\min(n, 2m), \min(n+1, 2m)\}$ if $2 \le m = n$ [7], and the minimum number of the typical rank of $m \times n \times p$ tensors with $3 \le m \le n$ is just $\min(p, mn)$ if $p \ge (m-1)n$ [6] over the real number field. In [4] we considered a generic form of $m \times n \times 3$

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tensors. Recently, Comon et al. [2] studied the minimum number of the typical rank of 3-tensors by using the Jacobian of the map

$$\{\boldsymbol{a}(r), \boldsymbol{b}(r), \boldsymbol{c}(r)\} \to T = \sum_{r=1}^{R} \boldsymbol{a}(r) \odot \boldsymbol{b}(r) \odot \boldsymbol{c}(r).$$

In contrast to that there may be more than one typical ranks over the real number field, we remark that a typical rank of n-tensors over the complex number field consists of just one number and thus it is called a generic rank. In this paper, we consider the smallest typical rank of n-tensors over the real number field. It is equal to the unique typical rank of n-tensors over the complex number field (cf. [5]).

A format $(p_1, \dots, p_n)$ is called "perfect" if $\max(p_1, \dots, p_n)$ is a typical rank of $T(p_1, \dots, p_n; \mathbb{R})$. Suppose that $2 \le p_1 \le p_2 \le p_3$. In [6], $p_1 \times p_2 \times p_3$ tensor is called "tall" if $p_1p_2 - p_2 < p_3 < p_1p_2$ and tall $p_1 \times p_2 \times p_3$ tensors have a unique typical rank p_3. Thus (p_1, p_2, p_3) is perfect if $p_1p_2 - p_2 < p_3 \le p_1p_2$. More generally, if $p_1p_2 - p_1 - p_2 + 2 \le p_3 \le p_1p_2$ then (p_1, p_2, p_3) is perfect (see [1, exercise 20.6, page 535]). We extend this result for n-tensors. Our main theorem is as follows.

Theorem 1.1 *Suppose that $n \ge 2$ and $2 \le p_1 \le \cdots \le p_n$. Let $q = p_1 \cdots p_n - (p_1 + \cdots + p_n) + n$. If $q \le p_{n+1} \le p_1 \cdots p_n$ then p_{n+1} is the smallest typical rank of $p_1 \times \cdots \times p_{n+1}$ tensors and $(p_1, \dots, p_{n+1})$ is perfect. Conversely if $(p_1, \dots, p_{n+1})$ is perfect then $q \le p_{n+1} \le p_1 \cdots p_n$.*

We show the theorem in the next section.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. First we give a range of typical ranks.

Lemma 2.1 *Let $2 \le p_1 \le p_2 \le \cdots \le p_{n+1} \le p_1 \cdots p_n$. A typical rank of $p_1 \times \cdots \times p_{n+1}$ tensors is greater than or equal to p_{n+1} and less than or equal to $p_1p_2 \cdots p_n$.*

Proof Let $A = (A_1; \cdots; A_{p_{n+1}})$ be an $p_1 \times \cdots \times p_{n+1}$ tensor, where A_j is a $p_1 \times \cdots \times p_n$ tensor for $j = 1, \dots, p_{n+1}$. Let consider the vector space V spanned by $A_1, \dots, A_{p_{n+1}}$. We denote by $f(A_j)$ a column vector given by flattening of A_j. Note that

$$\operatorname{rank}(A) \ge \operatorname{rank}(f(A_1), \dots, f(A_{p_{n+1}})) = \dim V.$$

If $\dim V < p_{n+1}$ then all p_{n+1}-minors of the matrix $\big(f(A_1) \ \ \cdots, f(A_{p_{n+1}})\big)$ are zero. Thus $\{(X_1; \cdots; X_{p_{n+1}}) \mid \dim\langle X_1, \dots, X_{p_{n+1}}\rangle = p_{n+1}\}$ is a Zariski open set in $T(p_1, \dots, p_{n+1}) \cong \mathbb{F}^{p_1 \cdots p_{n+1}}$. Thus a typical rank is greater than or equal to p_{n+1}.

In general $A = (a_{i_1 i_2 \dots i_n i_{n+1}})$ is described as a sum of $p_1 \cdots p_n$ rank one tensors

$$\boldsymbol{e}_{i_1}^{(1)} \odot \cdots \odot \boldsymbol{e}_{i_n}^{(n)} \odot (a_{i_1 \dots i_n 1}, \dots, a_{i_1 \dots i_n p_{n+1}}),$$

where $e_i^{(j)}$ is the i-th row vector of the $p_j \times p_j$ identity matrix. Thus $\operatorname{rank}(A) \le p_1 \cdots p_n$. ∎

Let $\varphi_1\colon \mathbb{R}^{p_1+\cdots+p_n} \to T(p_1,\dots,p_n)$ be a map defined by

$$\varphi_1(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) = \boldsymbol{a}_1 \odot \cdots \odot \boldsymbol{a}_n$$

and $\varphi\colon \mathbb{R}^{(p_1+\cdots+p_n)r} \to T(p_1,\dots,p_n)$ be a map defined by

$$\varphi(\boldsymbol{a}_1^{(1)},\dots,\boldsymbol{a}_n^{(1)},\dots,\boldsymbol{a}_1^{(r)},\dots,\boldsymbol{a}_n^{(r)}) = \sum_{h=1}^{r} \varphi_1(\boldsymbol{a}_1^{(h)},\dots,\boldsymbol{a}_n^{(h)}).$$

Put

$$\phi_1(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) := \begin{pmatrix} E_{p_1} \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_n \\ \boldsymbol{a}_1 \otimes E_{p_2} \otimes \cdots \otimes \boldsymbol{a}_n \\ \vdots \\ \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_{p_{n-1}} \otimes E_{p_n} \end{pmatrix} \tag{2.2}$$

for $\boldsymbol{a}_1 \in \mathbb{R}^{p_1}$, …, $\boldsymbol{a}_n \in \mathbb{R}^{p_n}$. Then the Jacobian $J(\varphi)$ of φ at

$$(\boldsymbol{a}_1^{(1)},\dots,\boldsymbol{a}_n^{(1)},\dots,\boldsymbol{a}_1^{(r)},\dots,\boldsymbol{a}_n^{(r)})$$

is given by

$$\begin{pmatrix} \phi_1(\boldsymbol{a}_1^{(1)},\dots,\boldsymbol{a}_n^{(1)}) \\ \vdots \\ \phi_1(\boldsymbol{a}_1^{(r)},\dots,\boldsymbol{a}_n^{(r)}) \end{pmatrix}.$$

If r is a typical rank of $T(p_1,p_2,p_3)$ then

$$\frac{p_1p_2p_3}{p_1+p_2+p_3-2} \le r \le \min(p_1p_2, p_1p_3, p_2p_3)$$

[3, 1]. This result also holds for n-tensors.

Proposition 2.3 *A typical rank of $p_1 \times \cdots \times p_n$ tensors is greater than or equal to*

$$\frac{p_1p_2\cdots p_n}{p_1+p_2+\cdots+p_n-n+1}$$

and less than or equal to

$$\min(p_2p_3\cdots p_n, p_1p_3\cdots p_n, \dots, p_1p_2\cdots p_{n-1}).$$

Proof Let consider the Segre embedding which is a map of projective spaces

$$RP^{p_1-1} \times \cdots \times RP^{p_n-1} \to RP^{p_1\cdots p_n-1}$$

induced by the tensor product map φ_1. The image $\mathrm{im}(\varphi_1)$ has dimension $p_1+p_2+\cdots+p_n-n$. Since $\{\boldsymbol{a}_1 \odot \dots \odot \boldsymbol{a}_n \mid \boldsymbol{a}_j \in \mathbb{R}^{p_j}\}$ is the affine cone of $\mathrm{im}(\varphi_1)$, it's dimension is $p_1+p_2+\cdots+p_n-n+1$. If r is a typical rank of $T(p_1,\dots,p_n)$, then $\dim T(p_1,\dots,p_n) \le r\dim(\mathrm{im}(\varphi_1))$ and thus

$$r \ge \frac{p_1\cdots p_n}{p_1+p_2+\cdots+p_n-n+1}.$$

∎

From now on, let $2 \le p_1 \le p_2 \le \cdots \le p_n$ and put $q = p_1 p_2 \cdots p_n - (p_1 + p_2 + \cdots + p_n) + n$. Suppose that $q \le p_{n+1} \le p_1 p_2 \cdots p_n$. By Lemma 2.1 it suffices to show that the Jacobian $J(\varphi)$ has full rank at some point.

Let S be a subset of

$$\{(k_1, \ldots, k_n) \mid 1 \le k_j \le p_j,\ j = 1, \ldots n\}$$

with cardinality p_{n+1} which contains

$$S_0 = \{(k_1, \ldots, k_n) \mid 1 \le k_j \le p_j,\ \#\{j \mid k_j = p_j\} \ne n - 1\}$$

and let $f\colon S \to \{1, 2, \ldots, p_{n+1}\}$ be a bijection.

We define maps $u_1, u_2, \ldots, u_n$ by $u_j(x_1, \ldots, x_n) = 0$ if $x_j = p_j$, $u_j(x_1, \ldots, x_n) = 1$ if $x_s = p_s$ for some $s \ne j$ and otherwise $u_j(x_1, \ldots, x_n) = x_j + 1$, for $j = 1, \ldots, n$.

We denote by $\boldsymbol{e}_j$ the jth row vector of the identity matrix. We put $\boldsymbol{a}_k^{(h)} \in \mathbb{R}^{p_h}$, $h = 1, \ldots, n+1$, as

$$\begin{aligned} \boldsymbol{a}_{f(k_1,\ldots,k_n)}^{(h)} &= \boldsymbol{e}_{k_h} + u_h(k_1, \ldots, k_n)\boldsymbol{e}_{p_h}, \quad 1 \le h \le n \\ \boldsymbol{a}_{f(k_1,\ldots,k_n)}^{(n+1)} &= \boldsymbol{e}_{f(k_1,\ldots,k_n)} \end{aligned}$$

for all $(k_1, \ldots, k_n) \in S$.

We denote the row vector $\boldsymbol{x}$ as $(x(k_1, \ldots, k_{n+1}))$ if

$$\boldsymbol{x} = \sum_{k_1,\ldots,k_{n+1}} x(k_1, \ldots, k_{n+1})\boldsymbol{e}_{k_1} \otimes \cdots \otimes \boldsymbol{e}_{k_{n+1}}.$$

Let $g\colon \mathbb{R}^{p_1 \cdots p_{n+1}} \to \mathbb{R}[x(1, \ldots, 1), \ldots, x(p_1, \ldots, p_{n+1})]$ be a map defined by

$$g\Big(\sum_{k_1,\ldots,k_{n+1}} h_{k_1,\ldots,k_{n+1}} \boldsymbol{e}_{k_1} \otimes \cdots \otimes \boldsymbol{e}_{k_{n+1}}\Big) = \sum_{k_1,\ldots,k_{n+1}} h_{k_1,\ldots,k_{n+1}} x(k_1, \ldots, k_{n+1}).$$

Note that g is linear, that is, it holds that

$$g(s_1\boldsymbol{y}_1 + s_2\boldsymbol{y}_2) = s_1 g(\boldsymbol{y}_1) + s_2 g(\boldsymbol{y}_2)$$

for $s_1, s_2 \in \mathbb{R}$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^{p_1 \cdots p_{n+1}}$. We abbreviate $\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n}$ to $\boldsymbol{e}(i_1, \ldots, i_n)$, $u_j(k_1, \ldots, k_n)$ to u_j, and $u_j(i'_1, \ldots, i'_n)$ to v_j. Then $x(i_1, \ldots, i_n) = g(\boldsymbol{e}(i_1, \ldots, i_n))$.

Put

$$\boldsymbol{z} = (\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_1^{(n+1)}, \ldots, \boldsymbol{a}_{p_{n+1}}^{(1)}, \ldots, \boldsymbol{a}_{p_{n+1}}^{(n+1)}).$$

We prepare three lemmas to show that the equation $J(\varphi(\boldsymbol{z}))\boldsymbol{x}^T = \boldsymbol{0}$ has no nonzero solution.

Lemma 2.4 *Let $n \ge 2$. Suppose that*

$$g((\boldsymbol{e}_{k_1} + \boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{k_n} + \boldsymbol{e}_{p_n})) = 0$$

for any $(k_1, \ldots, k_n) \in S_0 \smallsetminus \{(p_1, \ldots, p_n)\}$. Then it holds that

$$\begin{aligned} x(k_1, k_2, \ldots, k_n) = (-1)^{n-1}(&x(k_1, p_2, p_3, \ldots, p_n) + x(p_1, k_2, p_3, \ldots, p_n) \\ &+ \cdots + x(p_1, p_2, \ldots, p_{n-1}, k_n) + (n-1)x(p_1, p_2, \ldots, p_n)). \end{aligned}$$

Proof We show the assertion by induction on n. If $n = 2$ then the assertion

$$g(\boldsymbol{e}_{k_1} \otimes \boldsymbol{e}_{k_2}) = -g(\boldsymbol{e}_{k_1} \otimes \boldsymbol{e}_{p_2} + \boldsymbol{e}_{p_1} \otimes \boldsymbol{e}_{k_2}) - g(\boldsymbol{e}_{p_1} \otimes \boldsymbol{e}_{p_2})$$

follows from

$$(\boldsymbol{e}_{k_1} + \boldsymbol{e}_{p_1}) \otimes (\boldsymbol{e}_{k_2} + \boldsymbol{e}_{p_2}) = \boldsymbol{e}_{k_1} \otimes \boldsymbol{e}_{k_2} + (\boldsymbol{e}_{k_1} \otimes \boldsymbol{e}_{p_2} \boldsymbol{e}_{p_1} \otimes \boldsymbol{e}_{k_2}) + \boldsymbol{e}_{p_1} \otimes \boldsymbol{e}_{p_2}.$$

Put

$$W_n = \boldsymbol{e}(k_1, p_2, \dots, \boldsymbol{e}_{p_n}) + \boldsymbol{e}(p_1, k_2, p_3, \dots, \boldsymbol{e}_{p_n}) + \cdots + \boldsymbol{e}(p_1, \dots, \boldsymbol{e}_{p_{n-1}}, \boldsymbol{e}_{k_n})$$

for short. We have

$$\begin{aligned}(W_n + &n\boldsymbol{e}(p_1, \dots, p_n)) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}}) \\ &= \sum_{h=1}^{n} (\boldsymbol{e}(p_1, \dots, p_{h-1}, k_h, p_{h+1}, \dots, p_n, k_{n+1}) \\ &\qquad + \boldsymbol{e}(p_1, \dots, p_n) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}})) + W_n \otimes \boldsymbol{e}_{p_{n+1}} \\ &= 0.\end{aligned}$$

As the induction assumption, we assume that

$$g((\boldsymbol{e}_{k_1} + \boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{k_n} + \boldsymbol{e}_{p_n})) = 0$$

implies

$$g(\boldsymbol{e}(k_1, \dots, k_n)) = (-1)^{n-1} g(W_n + (n-1)\boldsymbol{e}(p_1, \dots, p_n))$$

for any $(k_1, \dots, k_n)$ and any $(p_1, \dots, p_n)$. Then we have

$$\begin{aligned}0 &= g((\boldsymbol{e}_{k_1} + \boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{k_n} + \boldsymbol{e}_{p_n}) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}})) \\ &= g((\boldsymbol{e}(k_1, \dots, k_n) + (-1)^n (W_n + (n-1)\boldsymbol{e}(p_1, \dots, p_n))) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}})) \\ &= g((\boldsymbol{e}(k_1, \dots, k_n) - (-1)^n \boldsymbol{e}(p_1, \dots, p_n)) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}})) \\ &= g(\boldsymbol{e}(k_1, \dots, k_{n+1}) + (-1)^{n-1} (W_n + (n-1)\boldsymbol{e}(p_1, \dots, p_n)) \otimes \boldsymbol{e}_{p_{n+1}} \\ &\qquad - (-1)^n \boldsymbol{e}(p_1, \dots, p_n) \otimes (\boldsymbol{e}_{k_{n+1}} + \boldsymbol{e}_{p_{n+1}})) \\ &= g(\boldsymbol{e}(k_1, \dots, k_{n+1}) - (-1)^n [W_{n+1} + n\boldsymbol{e}(p_1, \dots, p_{n+1})])\end{aligned}$$

Therefore the assertion holds for $n + 1$. ∎

Lemma 2.5 *We suppose that $v_1 = 1$ if $n = 1$. If*

$$g((\boldsymbol{e}_{i'_1} + v_1 \boldsymbol{e}_{p_1}) \cdots (e_{i'_n} + v_n \boldsymbol{e}_{p_n})) = 0$$

for any $1 \le i'_j \le p_j$, $j = 1, \dots, n$ such that $(i'_1, \dots, i'_n) \ne (p_1, \dots, p_n)$ then

$$g((e_{k_1} + u_1 \boldsymbol{e}_{p_1}) \cdots (e_{k_n} + v_n \boldsymbol{e}_{p_n})) = (u_1 - 1) \cdots (u_k - 1) x(p_1, \dots, p_n).$$

Proof We show the assertion by induction on n. If $n = 1$ then

$$\begin{aligned} g(e_{k_1} + u_1\boldsymbol{e}_{p_1}) &= g((e_{k_1} + u_1\boldsymbol{e}_{p_1}) - (e_{k_1} + v_1\boldsymbol{e}_{p_1})) \\ &= (u_1 - 1)x(p_1). \end{aligned}$$

As the induction assumption, we assume that the assertion holds for n and any $p_1, \ldots, p_n$. Putting $\beta = u_1(i_1, i_2, \ldots, k_{n+1})$, we have

$$\begin{aligned} & g((e_{k_1} + u_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (e_{k_{n+1}} + v_{n+1}\boldsymbol{e}_{p_{n+1}})) \\ &\quad = g((e_{k_1} + u_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (e_{k_n} + v_n\boldsymbol{e}_{p_n}) \otimes (e_{k_{n+1}} + \beta\boldsymbol{e}_{p_{n+1}})) \\ &\qquad\quad + (u_{n+1} - \beta)g((e_{k_1} + u_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (e_{k_n} + u_n\boldsymbol{e}_{p_n}) \otimes \boldsymbol{e}_{p_{n+1}})) \\ &\quad = (u_1 - 1)\cdots(u_n - 1)g(\boldsymbol{e}(p_1, \ldots, p_n) \otimes (\boldsymbol{e}_{k_{n+1}} + \beta\boldsymbol{e}_{p_{n+1}}))) \\ &\qquad\quad + (u_1 - 1)\cdots(u_n - 1)(u_{n+1} - \beta)g(\boldsymbol{e}(p_1, \ldots, p_n) \otimes \boldsymbol{e}_{p_{n+1}}) \\ &\quad = (u_1 - 1)\cdots(u_n - 1)g(\boldsymbol{e}(p_1, \ldots, p_n) \otimes \boldsymbol{e}_{k_{n+1}}) \\ &\qquad\quad + (u_1 - 1)\cdots(u_n - 1)u_{n+1}g(\boldsymbol{e}(p_1, p_2, \ldots, p_{n+1})) \\ &\quad = -1(u_1 - 1)\cdots(u_n - 1)x(p_1, p_2, \ldots, p_{n+1}) \\ &\quad + (u_1 - 1)\cdots(u_n - 1)u_{n+1}x(p_1, p_2, \ldots, p_{n+1}) \\ &\quad = (u_1 - 1)\cdots(u_{n+1} - 1)x(p_1, p_2, \ldots, p_{n+1}). \end{aligned}$$

We complete the proof. ∎

Lemma 2.6 *Suppose that $n = 2$, $2 \leq p_1 \leq p_2 \leq p_3$, $p_1p_2 - p_1 - p_2 + 3 \leq p_3 \leq p_1p_2$. Then the equation $J(\varphi(\boldsymbol{z}))\boldsymbol{x}^T = \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$.*

Proof The equation $J(\varphi(\boldsymbol{z}))\boldsymbol{x}^T = \boldsymbol{0}$ indicate

$$\begin{aligned} x(i_1', k_2, f(k_1, k_2)) + u_2x(i_1', p_2, f(k_1, k_2)) &= 0, \quad (2.7) \\ x(k_2, i_2', f(k_1, k_2)) + u_1x(p_1, i_2', f(k_1, k_2)) &= 0, \quad (2.8) \\ x(i_1, i_2, f(k_1, k_2)) + v_1x(p_1, i_2, f(k_1, k_2)) + v_2x(i_1, p_2, f(k_1, k_2)) & \\ +v_1v_2x(p_1, p_2, f(k_1, k_2)) &= 0, \quad (2.9) \end{aligned}$$

for $1 \leq i_1' \leq p_1$, $1 \leq i_2' \leq p_2$, and $(i_1, i_2), (k_1, k_2) \in S$. The equation (2.9) for $(i_1', i_2') = (p_1, p_2)$ is

$$x(p_1, p_2, f(k_1, k_2)) = 0, \quad (2.10)$$

Thus by (2.10), the equations (2.7) for $i_1' = p_1$ and (2.8) for $i_2' = p_2$ and (2.9) are

$$\begin{aligned} x(p_1, k_2, f(k_1, k_2)) &= 0 \quad (2.11) \\ x(k_1, p_2, f(k_1, k_2)) &= 0 \quad (2.12) \\ x(i_1, i_2, f(k_1, k_2)) + v_1x(p_1, i_2, f(k_1, k_2)) + v_2x(i_1, p_2, f(k_1, k_2)) &= 0 \quad (2.13) \end{aligned}$$

for $1 \leq i_1 < p_1$, $1 \leq i_2 < p_2$ and $(i_1, i_2), (k_1, k_2) \in S$. If $(k_1, k_2) = (p_1, p_2)$ then

$$x(i_1, i_2, f(p_1, p_2)) = 0$$

for $1 \leq i_1 < p_1$ and $1 \leq i_2 < p_2$ by (2.11), (2.12) and (2.13). Put together with (2.10), (2.11) and (2.12), we get

$$x(i'_1, i'_2, f(p_1, p_2)) = 0$$

for $1 \leq i'_1 \leq p_1$ and $1 \leq i'_2 \leq p_2$.

Now we show that $x(i'_1, i'_2, f(k_1, k_2)) = 0$ for $1 \leq i'_1 \leq p_1$, $1 \leq i'_2 \leq p_2$, $(k_1, k_2) \in S$ and $(k_1, k_2) \neq (p_1, p_2)$. Suppose that $(k_1, k_2) \neq (p_1, p_2)$. It follows from $(k_1, k_2) \in S$ that $k_1 < p_1$ and $k_2 < p_2$. By combining (2.7) for $i'_1 = i_1$, (2.11) and (2.13) for $i_2 = k_2$, we have

$$(u_2(i_1, k_2) - u_2(k_1, k_2))x(i_1, p_2, f(k_1, k_2)) \quad = \quad 0$$

for $1 \leq i_1 < p_1$. Thus

$$x(i_1, p_2, f(k_1, k_2)) = 0$$

for $1 \leq i_1 < p_1$, $i_1 \neq k_1$. Therefore $x(i'_1, p_2, f(k_1, k_2)) = 0$ for $1 \leq i'_1 \leq p_1$ by (2.10) and (2.12). Similarly by combining (2.8) for $i'_2 = i_2$, (2.12) and (2.13) for $i_1 = k_1$, we have

$$(u_1(k_1, i_2) - u_1(k_1, k_2))x(p_1, i_2, f(k_1, k_2)) \quad = \quad 0$$

which induces

$$x(p_1, i_2, f(k_1, k_2)) = 0$$

for $1 \leq i_2 < p_2$, $j \neq k_2$, and thus $x(p_1, i'_2, f(k_1, k_2)) = 0$ for $1 \leq i'_2 \leq p_2$ and $(k_1, k_2) \in S$ by (2.10) and (2.11). Thus by (2.13) again, we get $x(i_1, i_2, f(k_1, k_2)) = 0$ for $1 \leq i_1 < p_1$, $1 \leq i_2 < p_2$. Therefore $x(i'_1, i'_2, f(k_1, k_2)) = 0$ for $1 \leq i'_1 \leq p_1$, $1 \leq i'_2 \leq p_2$. Consequently we get $\boldsymbol{x} = \mathbf{0}$. ∎

Theorem 2.14 *The equation $J(\varphi(\boldsymbol{z}))\boldsymbol{x}^T = \mathbf{0}$ implies $\boldsymbol{x} = \mathbf{0}$ under the assumption in Theorem 1.1.*

Proof We consider the linear equation $J(\varphi(\boldsymbol{z}))\boldsymbol{x}^T = \mathbf{0}$. This equation is equivalent to

$$\psi_1(\boldsymbol{a}_k^{(1)}, \ldots, \boldsymbol{a}_k^{(n+1)})\boldsymbol{x}^T = \mathbf{0},\ 1 \leq k \leq p_n.$$

By (2.2), these equations indicate the following:

$$\begin{aligned}
g(\boldsymbol{e}_{i'_1} \otimes \boldsymbol{a}_k^{(2)} \otimes \boldsymbol{a}_k^{(3)} \otimes \cdots \otimes \boldsymbol{a}_k^{(n+1)}) &= 0, \\
g(\boldsymbol{a}_k^{(1)} \otimes \boldsymbol{e}_{i'_2} \otimes \boldsymbol{a}_k^{(3)} \otimes \cdots \otimes \boldsymbol{a}_k^{(n+1)}) &= 0, \\
&\vdots \\
g(\boldsymbol{a}_k^{(1)} \otimes \cdots \otimes \boldsymbol{a}_k^{(n-1)} \otimes \boldsymbol{e}_{i'_n} \otimes \boldsymbol{a}_k^{(n+1)}) &= 0, \\
g(\boldsymbol{a}_k^{(1)} \otimes \cdots \otimes \boldsymbol{a}_k^{(n-1)} \otimes \boldsymbol{a}_k^{(n)} \otimes \boldsymbol{e}_{i'_{n+1}}) &= 0.
\end{aligned}$$

for $1 \le k \le p_n$. In this proof, we always assume that i'_j is taken over $1, 2, \ldots, p_j$ for each $j = 1, \ldots, n$. Thus

$$\begin{aligned}
g((\boldsymbol{e}_{i'_1} \otimes (\boldsymbol{e}_{k_2} + u_2\boldsymbol{e}_{p_2}) \otimes \cdots \otimes (\boldsymbol{e}_{k_n} + u_n\boldsymbol{e}_{p_n}) \otimes \boldsymbol{e}_{f(k_1,\ldots,k_n)}) &= 0, \quad (2.15)\\
g((\boldsymbol{e}_{k_1} + u_1\boldsymbol{e}_{p_1}) \otimes \boldsymbol{e}_{i'_2} \otimes (\boldsymbol{e}_{k_3} + u_3\boldsymbol{e}_{p_3}) \otimes \cdots \otimes \boldsymbol{e}_{f(k_1,\ldots,k_n)}) &= 0, \quad (2.16)\\
&\vdots\\
g((\boldsymbol{e}_{k_1} + u_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{k_{n-1}} + u_{n-1}\boldsymbol{e}_{p_{n-1}}) \otimes \boldsymbol{e}_{i'_n} \otimes \boldsymbol{e}_{f(k_1,\ldots,k_n)}) &= 0, \quad (2.17)\\
g((\boldsymbol{e}_{i_1} + v_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{i_n} + v_n\boldsymbol{e}_{p_n}) \otimes \boldsymbol{e}_{f(k_1,\ldots,k_n)}) &= 0. \quad (2.18)
\end{aligned}$$

for any $(i_1, \ldots, i_n), (k_1, \ldots, k_n) \in S$.

We show the assertion by induction on n. The assertion for $n = 2$ holds by Lemma 2.6. We suppose that $n \ge 3$ and the assertion holds for $n - 1$ as the induction assumption.

By putting $(i'_1, \ldots, i'_n) = (p_1, \ldots, p_n)$, we get

$$x(p_1, \ldots, p_n, f(k_1, \ldots, k_n)) = 0 \tag{2.19}$$

for any $(k_1, \ldots, k_n) \in S$. Now let $k_n = p_n$. Put $f_1 = f(k_1, \ldots, k_{n-1}, p_n)$ for short. Then $u_1 = \cdots = u_{n-1} = 1$ and $u_n = 0$. By the n equations (2.15)-(2.17), the induction assumption yields us

$$x(i'_1, \ldots, i'_{n-1}, p_n, f_1) = 0 \tag{2.20}$$

for any $(k_1, \ldots, k_{n-1}, p_n) \in S$ and any $i'_1, \ldots, i'_{n-1}$. Then, by (2.18) we get

$$g((\boldsymbol{e}_{i_1} + v_1\boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{i_{n-1}} + v_{n-1}\boldsymbol{e}_{p_{n-1}}) \otimes \boldsymbol{e}_{i_n} \otimes \boldsymbol{e}_{f_1}) = 0 \tag{2.21}$$

for all $(i_1, \ldots, i_n) \in S$. This equation and (2.17) indicate

$$x(p_1, \ldots, p_{n-1}, i_n, f_1) = 0 \tag{2.22}$$

by Lemma 2.5 if $i_n < p_n$. Suppose that $i_n < p_n$. In the equation (2.21) we put $i_j = p_j$ for $n - 2$ numbers j's with $j < n$ and get

$$x(i_1, p_2, \ldots, p_{n-1}, i_n, f_1) = \cdots = x(p_1, \ldots, p_{n-2}, i_{n-1}, i_n, f_1) = 0$$

for $1 \le i_j < p_j$, $j = 1, \ldots, n$, and thus

$$x(i'_1, p_2, \ldots, p_{n-1}, i_n, f_1) = \cdots = x(p_1, \ldots, p_{n-2}, i'_{n-1}, i_n, f_1) = 0 \tag{2.23}$$

for any $i'_1, \ldots, i'_n$ by (2.22). In the equation (2.21) we put $i_j = p_j$ for $n - 3$ numbers j's and get

$$x(i_1, i_2, p_3, \ldots, p_{n-1}, i_n, f_1) = \cdots = x(p_1, \ldots, p_{n-3}, i_{n-2}, i_{n-1}, i_n, f_1) = 0$$

for $1 \le i_j < p_j$, $j = 1, \ldots, n$, and thus

$$x(i'_1, i'_2, p_3, \ldots, p_{n-1}, i_n, f_1) = \cdots = x(p_1, \ldots, p_{n-3}, i'_{n-2}, i'_{n-1}, i_n, f_1) = 0$$

by (2.23). And go on, finally we get

$$x(i'_1, \dots, i'_{n-1}, i_n, f_1) = 0$$

for any $i'_1, \dots, i'_{n-1}$ and any $1 \le i_n < p_n$ and then by (2.20)

$$x(i'_1, \dots, i'_{n-1}, i'_n, f_1) = 0$$

for any $i'_1, \dots, i'_n$. If we consider the similar argument for j instead of n, we have

$$x(i'_1, \dots, i'_n, f(k_1, \dots, k_n)) = 0$$

for any $i'_1, \dots, i'_n$ and any $(k_1, \dots, k_n) \in S$ with $k_j = p_j$ for some j.

To complete the proof, it suffices to show that

$$x(i'_1, \dots, i'_n, f(k_1, \dots, k_n)) = 0$$

for any $i'_1, \dots, i'_n$ and any $(k_1, \dots, k_n) \in S$ with $k_j < p_j$ for each j. Let $f_2 = f(k_1, \dots, k_n)$ for short. By putting $i_n = p_n$ in (2.18), we get

$$g((\boldsymbol{e}_{i_1} + \boldsymbol{e}_{p_1}) \otimes \cdots \otimes (\boldsymbol{e}_{i_{n-1}} + \boldsymbol{e}_{p_{n-1}}) \otimes \boldsymbol{e}_{p_n} \otimes \boldsymbol{e}_{f_2}) = 0$$

for $(i_1, \dots, i_{n-1}, p_n) \in S$. By Lemma 2.4, we have

$$\begin{aligned} 0 &= g((\boldsymbol{e}(i_1, p_2, \dots, p_{n-1}) + \cdots + \boldsymbol{e}(p_1, \dots, p_{n-2}, i_{n-1}) \\ &\qquad + (n-2)\boldsymbol{e}(p_1, \dots, p_{n-1})) \otimes \boldsymbol{e}(p_n, f_2)) \\ &= g((\boldsymbol{e}(i_1, p_2, \dots, p_{n-1}) + \cdots + \boldsymbol{e}(p_1, \dots, p_{n-2}, i_{n-1})) \otimes \boldsymbol{e}(p_n, f_2)). \end{aligned}$$

Thus

$$g((\boldsymbol{e}(i_1, p_2, \dots, p_n) + \cdots + \boldsymbol{e}(p_1, \dots, p_{n-2}, i_{n-1}, p_n)) \otimes \boldsymbol{e}_{f_2}) = 0.$$

Similarly, for each $j = 1, \dots, n-1$, by putting $i_j = p_j$ in (2.18), we get

$$\begin{aligned} g((\boldsymbol{e}(p_1, i_2, p_3, \dots, p_n) + \cdots + \boldsymbol{e}(p_1, \dots, p_{n-1}, i_n)) \otimes \boldsymbol{e}_{f_2}) &= 0, \\ &\vdots \\ g((\boldsymbol{e}(i_1, p_2, \dots, p_n) + \cdots + \boldsymbol{e}(p_1, \dots, p_{p-3}, p_{i-2}, p_{n-1}, p_n) \\ +\boldsymbol{e}(p_1, \dots, p_{n-1}, i_n)) \otimes \boldsymbol{e}_{f_2}) &= 0. \end{aligned}$$

Since

$$\left| \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} - E_n \right| = (-1)^{n-2}(n-1),$$

we have

$$x(i_1, p_2, \dots, p_n, f_2) = \cdots = x(p_1, \dots, p_{n-1}, i_n, f_2) = 0$$

for $1 \le i_j < p_j$, $j = 1, \dots, n$, and then

$$x(i'_1, p_2, \dots, p_n, f_2) = \cdots = x(p_1, \dots, p_{n-1}, i'_n, f_2) = 0$$

for all $i'_1, \ldots, i'_n$, since $x(p_1, p_2, \ldots, p_n, f_2) = 0$. By putting $i'_j = p_j$ for $n-2$ numbers j's in the equation (2.18) we get

$$x(i_1, i_2, p_3, \ldots, p_n, f_2) = \cdots = x(p_1, \ldots, p_{n-2}, i_{n-1}, i_n, f_2) = 0$$

for $1 \le i_j < p_j$, $j = 1, \ldots, n$, and then

$$x(i'_1, i'_2, p_3, \ldots, p_n, f_2) = \cdots = x(p_1, \ldots, p_{n-2}, i'_{n-1}, i'_n, f_2) = 0$$

for all $i'_1, \ldots, i'_n$. And so on, we finally get

$$x(i'_1, \ldots, i'_n, f_2) = 0$$

for all $i'_1, \ldots, i'_n$. We complete the proof. ∎

Now we show Theorem 1.1.

Proof of Theorem 1.1 Let r be a typical rank of $p_1 \times \cdots \times p_{n+1}$ tensors. Then $p_{n+1} \le r \le p_1 p_2 \cdots p_n$ by Lemma 2.1. In particular, note that any integer less than p_{n+1} is not a typical rank. Since $p_{n+1} \ge q$, it holds that p_{n+1} is a typical rank by Theorem 2.14.

Conversely suppose that p_{n+1} is a typical rank of $p_1 \times \cdots \times p_{n+1}$ tensors. By Proposition 2.3,

$$p_{n+1} \ge \frac{p_1 \cdots p_{n+1}}{p_1 + \cdots + p_{n+1} - n}$$

which implies that $p_{n+1} \ge q$, and also, a typical rank is less than or equal to $p_1 \cdots p_n$. Thus $p_{n+1} \le p_1 \cdots p_n$. We complete the proof. ∎

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